Control of respiratory system using Volterra series

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This is a theoretical study of the control of artificial respiration for the maintenance of alveolar carbon dioxide concentration within a desired range of value, eliminating the effect of irregular metabolic rate change. Three different methods using Volterra series are applied to the design of control systems of artificial respiration, which are based on only the description of input–output relation of controlled objects by discrete-time Volterra series. They are intended to be used selectively according to different clinical purposes and the obtainable amount of knowledge of the controlled respiratory system. The first control system is based on an input-reproduction method. It requires almost perfect knowledge about the respiratory system including the effect of change of metabolic rate. The second is based on a partial model matching method. This requires only a limited knowledge of the respiratory system supposing that the effect of change of metabolic rate cannot be recognised at all. The last approach is based on an adaptive control method and does not require any knowledge about the respiratory system except for rough knowledge about its structure.

Keywords: Artificial respiration, Volterra series, synthesis of control system, discrete-time system.

1. Introduction

Recent improvements in respirators have made it possible to control the level of arterial alveolar CO_2 concentration according to different clinical purposes. In this connection, there have been proposed several methods of controlling the regulation of ventilation rate for the maintenance of alveolar CO_2 concentration within an appropriate range of value against the effect of irregular change of metabolic rate. However, conventional control systems have been designed on the basis of mathematical models obtained by linearisation of the non-linear characteristics of the respiratory system (Holloman *et al,* 1968; Noshiro, 1984). Therefore, there can theoretically always be an inevitable limitation in their capabilities caused by their non-linearities in practical use.

In this study, a Volterra series is adopted as a mathematical model for the description of the respiratory system, because it has generality for the representation and theoretical treatment of analytical non-linear systems (Volterra, 1959; Barrett, 1965). The aim is to obtain three different control systems for the regulation of alveolar CO_2 concentration according to the clinical demands to be expected and the amount of knowledge about the respiratory system, using only the information about input and output without any consideration of its internal state.

The first control system is based on an input-reproduction method (Wakamatsu, 1982) and requires almost perfect knowledge about the respiratory system including the effect of change of metabolic rate. The basic idea of this control system, which possesses characteristics of a rapid response, is the realisation of a desired value on the basis of the reproduction of change of metabolic rate as an undesirable disturbance and the compensation of its effect on alveolar CO_2 concentration. The desired value is given by an output from a reference model which is considered to reflect clinically desirable characteristics, or by an equilibrium state reference.

The second system is based on a partial model-matching method with the same reference model (Kitamori, 1979; Wakamatsu and Kitamori, 1984). This requires only a limited knowledge of the respiratory system in the circumstance when the effect of change of metabolic rate cannot be recognised at all. Its basic idea is to synthesise control systems on the basis of an imperfect knowledge of controlled objects, recognising that perfect knowledge can rarely be obtained in actual systems, even if appropriate methods are applied in system identification (Billings, 1980). This method enables us to make the dynamics of control systems flexible according to the control requirements and assures their robustness against imperfect recognition of controlled objects and/or their parameter changes.

The last system is based on an adaptive control method (Åstrom, 1983; Landau and Tomizuka, 1981) with the same reference model as used in the previous two cases or an equilibrium state reference. It does not require any knowledge about the respiratory system except for approximate knowledge about the structure

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of a controlled object. This enables us to control respiration to have an appropriate value or to show the desirable dynamic characteristics given beforehand by the physician, in cases when respiration must be artificially controlled in emergency clinics, even although the approximate parameters of the patient cannot be known on the spot for the control system synthesis.

2. Description of respiratory system by discrete-time Volterra series

Many different descriptions of respiratory function have been proposed, depending on the respiratory phenomena concerned. The derived models range from gas exchange models in erythrocyte to neuron network models of the regulation of respiratory function. However, it is important to take a broad view of the dynamics of the respiratory system resulting from the physiological connection of the functions of major components – alveolar-arterial and venous-tissue compartments which are essentially controlled by chemical inorganic substances – whereas any other regulation (e g, neural or endocrine) is not taken into consideration. That is, the controlled object is considered to consist of the above functional units, and the diaphragm and intercostal muscles which are actuated in response to commands from the pons and medulla oblongata with feedback information from the sensors of the aortic and carotid bodies. When subjected to artificial respiration, the above respiratory control function is forced to change according to the control law generated from the artificial controller, although it would work according to the physiological control law*.

For the purpose of the theoretical study of controlling respiration, the respiratory system is here assumed to be described by a non-linear input–output relation, through which unknown metabolic rate and air ventilation rate affect alveolar CO_2 concentration without subharmonic non-linearities in the frequency domain. The Volterra series is known as an appropriate mathematical model for the description of such a non-linear relation including the linear relation†. An equilibrium state of alveolar CO_2 concentration x^0 is given by the constant (air) ventilation rate u^0 and metabolic rate m^0. Let the deviations from x^0, u^0 and m^0 be denoted by x, u and m respectively and further be discretised with respect to time $p\tau$ (abbreviated hereafter p) as X_p, U_p and M_p, then the continuous function of the respiratory system is represented by discrete-time Volterra series as follows‡:

$$X_p=\sum_{i=0}^{p} h_i^u U_{p-i}+\sum_{i=0}^{p} h_i^m M_{p-i}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p} h_{ij}^{uu} U_{p-i}U_{p-j}+\sum_{i=0}^{p}\sum_{j=i}^{p} h_{ij}^{um} U_{p-i}M_{p-j}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p} h_{ij}^{mm} M_{p-i}M_{p-j}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{uuu} U_{p-i}U_{p-j}U_{p-k}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{uum} U_{p-i}U_{p-j}M_{p-k}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{umm} U_{p-i}M_{p-j}M_{p-k}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{mmm} M_{p-i}M_{p-j}M_{p-k}+\ldots \qquad \ldots (1)$$

In order to synthesise control systems to make an output follow-up the change in desired value in later sections, the transfer function with denominator series expression indicated by equation (2) is adopted as a reference model W, which is characterised by recommended parameters given by the set below§.

$$W(s)=1/(\beta_0+\beta_1\sigma s+\beta_2\sigma^2 s^2+\beta_3\sigma^3 s^3+\ldots) \qquad \ldots (2)$$
$$\{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \ldots\}$$
$$=\{1, 1, 0.5, 0.15, 0.03, 0.003, \ldots\}$$

3. Synthesis of control system of respiration based on the input-reproduction method

3.1. Knowledge about the respiratory system

At first, a control system of a respiration with a rapid response is discussed. This is a form of feedforward control method, which directly gives a desired value, based on the reproduction of disturbance (metabolic rate M_p) regarded as an input to the respiratory system and on the compensation of its effect on the controlled output (alveolar CO_2 concentration). The relations of changes of metabolic rate, ventilation rate and their interaction to alveolar CO_2 concentration change are assumed to be almost perfectly recognised excluding the non-linear terms of order higher than 3, although the respiratory system has been assumed to be represented by an infinite Volterra series. That is, knowledge of the respiratory system is assumed to be limited as given by equation (3).‖

$$X_p=\sum_{i=0}^{p} h_i^u U_{p-i}+\sum_{i=0}^{p} h_i^m M_{p-i}+\sum_{i=0}^{p}\sum_{j=i}^{p} h_{ij}^{uu} U_{p-i}U_{p-j}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p} h_{ij}^{um} U_{p-i}M_{p-j}+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{uuu} U_{p-i}U_{p-j}U_{p-k}$$
$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{uum} U_{p-i}U_{p-j}M_{p-k} \qquad \ldots (3)$$

*The control is performed by a proportional control law to CO_2 concentration; also to O_2 concentration when it sinks to an extreme level.

†It is well known that the analytical non-linear function can be expanded as a Volterra series (Volterra, 1959; Barrett, 1965).

‡A similar equation to equation (1) is required for the O_2 concentration change. This function is not described here, because it is not essential for the discussion of the control of respiration. Superscripts u, m mean non-linear degrees of contributions of changes of ventilation rate and metabolic rate to alveolar CO_2 concentration by their combinations and numbers.

§This is a normalised transfer function with respect to direct current gain and time scale. The models with different truncations of parameter series with the recommended values of the order higher than 2 show similar step responses with desirable characteristics from the viewpoint of control, stability and time of response in a variety of applications (Kitamori, 1979).

‖In fact, the dynamics of a respiratory system given by differential equation (17) can be sufficiently well approximated by a finite Volterra series with non-linearities of the order not higher than 3. This is verified comparing dynamic characteristics of the third order Volterra series with that of equation (17) within the physiologically appropriate range of inputs ($U_p \leqq 2u^0$) used for the control of respiration.

3.2. Principle and design of a control system (Wakamatsu, 1982, 1985a)

The change of metabolic rate, which is difficult to measure, is reproduced by the input-observer* and its effect on the alveolar CO_2 concentration change is theoretically calculated from equation (3) using the reproduced change of metabolic rate with its extrapolation and past change in ventilation rate used as a controlling input. For compensation of the effect of metabolic rate change on the output and for the follow-up control of the output to a desired value, the necessary controlling input value (ventilation rate change U_p) is provided from a dynamical compensator. The control system of respiration is synthesised as illustrated in Fig 1, where the respiratory system, an input-observer and a dynamical compensator are denoted by H, K_m and C_u, respectively. An input-observer can be synthesised as a system, which uniquely reproduces the difficult to observe change of metabolic rate from the information of the observable controlled value (alveolar CO_2 concentration change X_p) and known input (ventilation rate change U_p). As the respiratory system has been represented by the third order Volterra series with a deficit of some non-linear terms, its input-observer is synthesised as described in equation (4) on the basis of a slight modification of the ones given in the previous studies (Wakamatsu, 1981, 1982, 1985a).

$$M_p=-b_p/a_p\dagger \qquad \ldots (4)$$

The existence of the solution of the input-observer is guaranteed at any sampling time p by the condition $a_p\neq 0$, which results from $h_{000}^{uum}\neq 0$ and $(h_{00}^{um})^2<4(h_0^m)(h_{000}^{uum})$ for system parameters given as standard values in Section 6.1‡. As metabolic rate change M_p is reproduced with no delay§ by the input-observer, the dynamic compensator C_u consists of the 1-point extrapolator and functional generator described by equation (5) as indicated in Fig A-1.

$$A_{p+1}(U_{p+1})^3+B_{p+1}(U_{p+1})^2+C_{p+1}(U_{p+1})+D_{p+1}=0 \qquad \ldots (5)$$

This subsystem generates the ventilation rate change U_{p+1} to be used as a controlling input, which is calculated from the known input series $\{U_p\}$ and reproduced input series $\{M_p\}$ with estimated $\hat{M}_{p+1}$ so that the alveolar CO_2 concentration change X_{p+1} is realised to be a desired value $X_{d,p+1}$ at sampling time $p+1$. Because the respiratory system represented by equation (3) has no delay, $A_p=h_{000}^{uuu}(\neq 0)$ is also obtained. This ensures that there exists at least one real solution for the controlling input, because equation (5) is a third order algebraic equation. If three real solutions exist, the minimum distance solution to that at the previous time or the minimum absolute value solution is selected as a controlling input at the next sampling time.

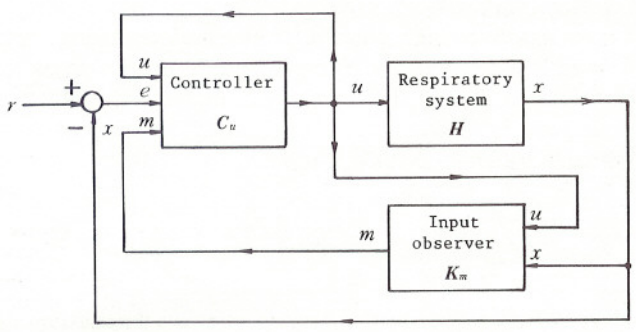

Fig 1 Block diagram of non-linear respiratory control system based on the input-reproduction method (continuous-time description). u: controlling input (air ventilation rate), x: controlled value (alveolar CO_2), m: disturbance (metabolic rate), r: desired value, e: controlled deviation.

4. Synthesis of control system based on the partial model-matching method

4.1. Knowledge about the respiratory system

The dynamic characteristics of the respiratory system have been originally described by equation (1), which is an infinite Volterra series. However, its perfect knowledge cannot be obtained in practice. Considering such practical conditions, non-linear terms of order higher than 3 are assumed to be unrecognisable. It is also assumed that the relation of metabolic rate change including its interaction with ventilation rate change to the alveolar CO_2 concentration change cannot be obtained, because metabolic rate change itself is hard to observe. The function of the respiratory system can then be described only by the following relation:

$$X_p=\sum_{i=0}^{p}h_i^u U_{p-i}+\sum_{i=0}^{p}\sum_{j=i}^{p}h_{ij}^{uu}U_{p-i}U_{p-j}$$

$$+\sum_{i=0}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p}h_{ijk}^{uuu}U_{p-i}U_{p-j}U_{p-k} \qquad \ldots (6)$$

That is, a respiratory system has been recognised only as a system of the third order Volterra series with an input of ventilation rate change and an output of alveolar CO_2 concentration change where metabolic rate change is not taken into consideration.

4.2. Principle and design of a control system (Wakamatsu and Kitamori, 1984; Wakamatsu, 1985b)

With the imperfect knowledge given by equation (6) about the respiratory system represented by equation (1), a control system is here synthesised in the form of feedback compensation, whose dynamics follow-up the

*This has been proposed as an inverse system of Volterra type non-linear system for the reproduction of its input series (Wakamatsu, 1981).

†This is precisely explained in Appendix A1.

‡The physiologically proper average values have been adopted. a_p takes a minimum value 0.234 [vol% min/l] and varies from 84–104% from its nominal value, when the main parameters V_1, V_2, η and Q concerning equation (17) change within the range of ±20%. The coefficients which characterise a_p can be readily measured, because they are located near the zero of the time co-ordinate (Wakamatsu, 1985a).

§'Delay' has been defined in Wakamatsu (1981), which corresponds to 'dead time' in linear system theory. The delay resulting from integration in the ordinary sense is naturally included in each general impulse response (Volterra kernel) itself – i e, there is one sampling delay in obtaining an output for any input.

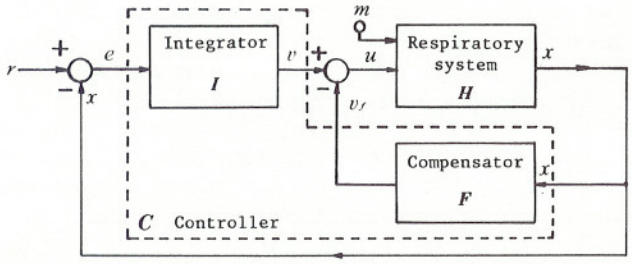

Fig 2 Block diagram of non-linear respiratory control system based on the partial model matching method (continuous-time description). u: controlling input (air ventilation rate), x: controlled value (alveolar CO_2), m: disturbance (metabolic rate), r: desired value, e: controlled deviation, v: output from integrator I, v_f: output from feedback compensator.

characteristics of a given reference model and whose steady-state position error is zero. The control system consists of a controlled respiratory system H and a controller C as depicted in Fig 2. The variables u, x, r, e are the input, output, desired value and controlled deviation. The variables v and v_f are the outputs from integrator I and feedback compensator F, respectively. As the design of the control system implies a realisation of characteristics $x=W(r)$, which is given by a reference model W, the controller C is synthesised as described by equation (7), taking into account the operational property of a Volterra series (Wakamatsu and Kitamori, 1984) with unit operator E. Namely, the controlling input u is given by

$$u=v-v_f \qquad \ldots (7)$$

where

$$v=I(e)=KD^{-1}(e)$$

$$v_f=F(x)=(KD^{-1}(W^{-1}-E)-H^{-1})(x)^*$$

Equation (7) implies that the controller is synthesised from the inverse system H^{-1} of an obtainable model of the respiratory system H and $(W^{-1}-E)$ obtained from a given reference model W and unit operator E. That is, a perfect model matching should be possible, if a perfect inverse system could be synthesised from perfect knowledge of the respiratory system. However, it is, of course, not possible, because the mathematical model represents only the partial characteristics of an actual respiratory system. It is, however, noted that a controlled object has been theoretically linearised in the perfect model matching under the condition that no metabolic rate change exists, because the transfer function $X(s)/V(s)$ for an input v and an output x can be written as

$$X(s)/V(s)=K^{-1}\sigma^{-1}(\beta_1+\beta_2\sigma s+\beta_3\sigma^2 s^2+\beta_4\sigma^3 s^3+\ldots)^{-1} \qquad \ldots (8)$$

This means that, if a perfect inverse of a perfect mathematical model of the respiratory system should exist, the poles of the linearised system could be assigned by selecting appropriate parameters so that the dynamic characteristics such as stability and time of response are improved. The proposed control system requires, as mentioned above, an inverse system on the basis of limited knowledge of the respiratory system. It is for the reproduction of the input U_p through observation of the output X_p of the respiratory system, under the condition that any metabolic rate change is not taken into consideration during the operation of the control system. The inverse system is adopted from the one proposed as a truncated form whose original has been given as an infinite series (Schetzen, 1976). This truncated inverse system is applied to the synthesis of the control system under some restriction of its input amplitude.

Here, let the mathematical description given by equation (6) be rewritten as $H_{(3)}=\{H_1, H_2, H_3\}$ using Volterra operators. If the inverse of equation (6) designated by K is expanded using a Volterra series and is approximated as $K_{(q)}=\{K_1, K_2, K_3, \ldots, K_q\}$, each operator of $K_{(q)}$ is then described as follows:

$$K_1=H_1^{-1}$$

$$K_2=-H_1^{-1}H_2H_1^{-1} \qquad \ldots (9)$$

$$K_3=-H_1^{-1}(H_3-H_2[E+H_1^{-1}H_2]+H_2H_1^{-1}H_2+H_2)H_1^{-1}$$

For the design of the control system, the smallest possible parameter σ and largest possible absolute value of K are determined numerically as the optimal ones under the condition that its response would not fall into disorder†, where account should be taken of the fact that the maximum amplitude of the input may not be out of the condition of convergence of the inverse system (Schetzen, 1976; Wakamatsu and Kitamori, 1984).

5. Synthesis of a control system of respiration based on an adaptive control method

5.1. Model obtained from knowledge about the respiratory system

As change in metabolic rate is hard to measure, its contribution to alveolar CO_2 concentration change is also assumed to be not known as in the previous section. Here, the contribution of the unknown disturbance to alveolar CO_2 concentration change is regarded as resulting from parameter deviation. As for the contribution of ventilation rate, only the finite order non-linear structure is assumed, although its infinite order non-linear contribution should be taken into account including the effect of its interaction with unknown metabolic rate change on the alveolar CO_2 concentration change. In order to synthesise a model reference adaptive control system, the following non-linear moving average model derived directly from a discrete-time Volterra series is introduced as an appropriate mathematical model‡:

$$X_p=\sum_{i=0}^{r_1} h_i^u U_{p-i}+\sum_{i=0}^{r_2}\sum_{j=i}^{r_2} h_{ij}^{uu} U_{p-i}U_{p-j} +\sum_{i=0}^{r_3}\sum_{j=i}^{r_3}\sum_{k=j}^{r_3} h_{ijk}^{uuu} U_{p-i}U_{p-j}U_{p-k}+\ldots \qquad \ldots (10)$$

*D is a differential operator s when a continuous-time system is considered, and a shift operator z when a discrete-time system is considered.

†In general, a control system has a shorter response time for a smaller σ and a shorter settling time for a larger absolute value of K.

‡The non-linear moving average model has been proposed as an extended form of discrete-time Volterra series for a non-linear system identification problem (Wakamatsu, 1985c).

where $r_c(c=1, 2, \ldots)$ indicates the memory length of present and past input data in the cth order non-linear term, which contributes to alveolar CO_2 concentration change.

Obtainable information about the respiratory system is regarded only as knowledge about its non-linear order and memory lengths of input concerning linear and non-linear terms, which correspond to the 'non-linear structure' of a controlled object.

The mathematical structure of the respiratory system is assumed to be given by equation (10) with non-linearity of order 3 and $r_1=r_2=r_3=2$*.

5.2. Principle and design of a control system (Landau and Tomizuka, 1981; Wakamatsu, 1986)

For the preparation of the design of an adaptive control system, the respiratory system is formulated by parametric representation as

$$X_p=f(U_p)+\theta_0^T\zeta_{0,p-1} \quad \ldots (11)$$

where $f(U_p)=\alpha_3(U_p)^3+\alpha_2(U_p)^2+\alpha_1(U_p)$ with a parameter vector θ_0 and a state vector $\zeta_{0,p-1}$ constructed from the input data†.

On the basis of the parametric representation given by Eqn (11), an adaptive controlling input and a parameter regulation law are derived as follows:

Let an output error be first defined as $e_p=X_p-X_p^M$, where X_p^M is an output from a reference model or a given sequential data series, then

$$e_p=f(U_p)+\theta_0^T\zeta_{0,p-1}-X_p^M \quad \ldots (12)$$

The controlling input U_p to the respiratory system is determined using estimated parameters $\hat{\theta}_{0,p-1}$ at previous sampling time $p-1$ so that $e_p=0$ is satisfied, namely,

$$\hat{f}_{p-1}(U_p)+\hat{\theta}_{0,p-1}^T\zeta_{0,p-1}-X_p^M=0‡ \quad \ldots (13)$$

Equation (13) is a third order algebraic equation. As $\hat{\alpha}_3=\hat{h}_{000}^{uuu}(\neq 0)$ holds good for the pertinent system which has no delay, the controlling input U_p can be obtained as a real solution. Referring to equation (12), the output error model is described using estimated parameters as

$$\hat{e}_p=\hat{f}_{p-1}(U_p)+\hat{\theta}_{0,p-1}^T\zeta_{0,p-1}-X_p^M \quad \ldots (14)$$

Then, the error equation is written using equations (12) and (14) as

$$\varepsilon_p=e_p-\hat{e}_p=\phi_p^T\xi_p \quad \ldots (15)$$

where

$$\phi_p^T=[\theta^T-\hat{\theta}_{p-1}^T]$$

$$\theta^T=[\theta_0^T, h_0^u, \theta_1^T, h_{00}^{uu}, \theta_2^T, h_{000}^{uuu}]$$

$$\xi_p^T=[\zeta_{0,p-1}^T, U_p, \zeta_{1,p-1}^T U_p, (U_p)^2, \zeta_{2,p-1}^T(U_p)^2, (U_p)^3]$$

If parameters are regulated according to equation (16),

$$\hat{\theta}_p=\hat{\theta}_{p-1}+\Gamma_{p-1}\xi_p\varepsilon_p \quad \ldots (16)$$

where

$$\varepsilon_p=(X_p-\hat{\theta}_{p-1}^T\xi_p)/(1+\xi_p^T\Gamma_{p-1}\xi_p)$$

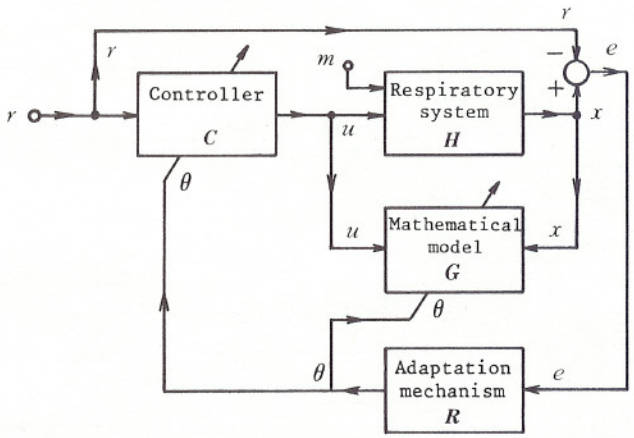

Fig 3 Block diagram of non-linear respiratory control system based on the adaptive control method (continuous-time description). u: controlling input (air ventilation rate), x: controlled value (alveolar CO_2), m: disturbance (metabolic rate), r: desired value, e: controlled deviation (output error). Slanting arrows denote the regulation of parameters belonging to pertinent subsystems.

and the gain Γ_p is given by

$$\Gamma_p^{-1}=\lambda_{1,p}\Gamma_{p-1}^{-1}+\lambda_{2,p}\xi_p\xi_p^T$$

$$(0<\lambda_{1,p}\leqq 1,\ 0\leqq\lambda_{2,p}<2,\ \Gamma_0>0),$$

the output error ε_p converges to zero, because equation (15) is linear with respect to the parameters§. The outline of this control system is depicted in Fig 3.

6. Characteristics of control systems

6.1. Description of a mathematical model regarded as an actual respiratory system

Simulation experiments are carried out to clarify the features of control systems of the respiratory system. For this purpose, a mathematical model is required which represents the main characteristics of the respiratory regulation system resulting from the physiological connection of essential components. The physiological controller consists of the diaphragm and intercostal muscles which are actuated by the order of the pons and medulla oblongata with chemical feedback information from the sensors. Nevertheless, this control function is forcedly changed under artificial respiration by the control law generated from the artificial controller mainly subject to CO_2 concentration, although it would work according to the physiological control law. From this point of view, the respiratory system can be assumed to

*The memory length of the input series in each term can be different according to the characteristics of non-linearity of a system.

†The details of the vectors are explained in Appendix A4.

‡The notation ^ over the head of a parameter implies an estimated value of a system parameter at sampling time p. $\hat{f}_{p-1}(U_p)$ implies $f(U_p)$ with estimated parameters $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$ instead of α_1, α_2, and α_3.

§The convergence of the output error is proved by the method given by Landau (Landau and Tomizuka, 1981).

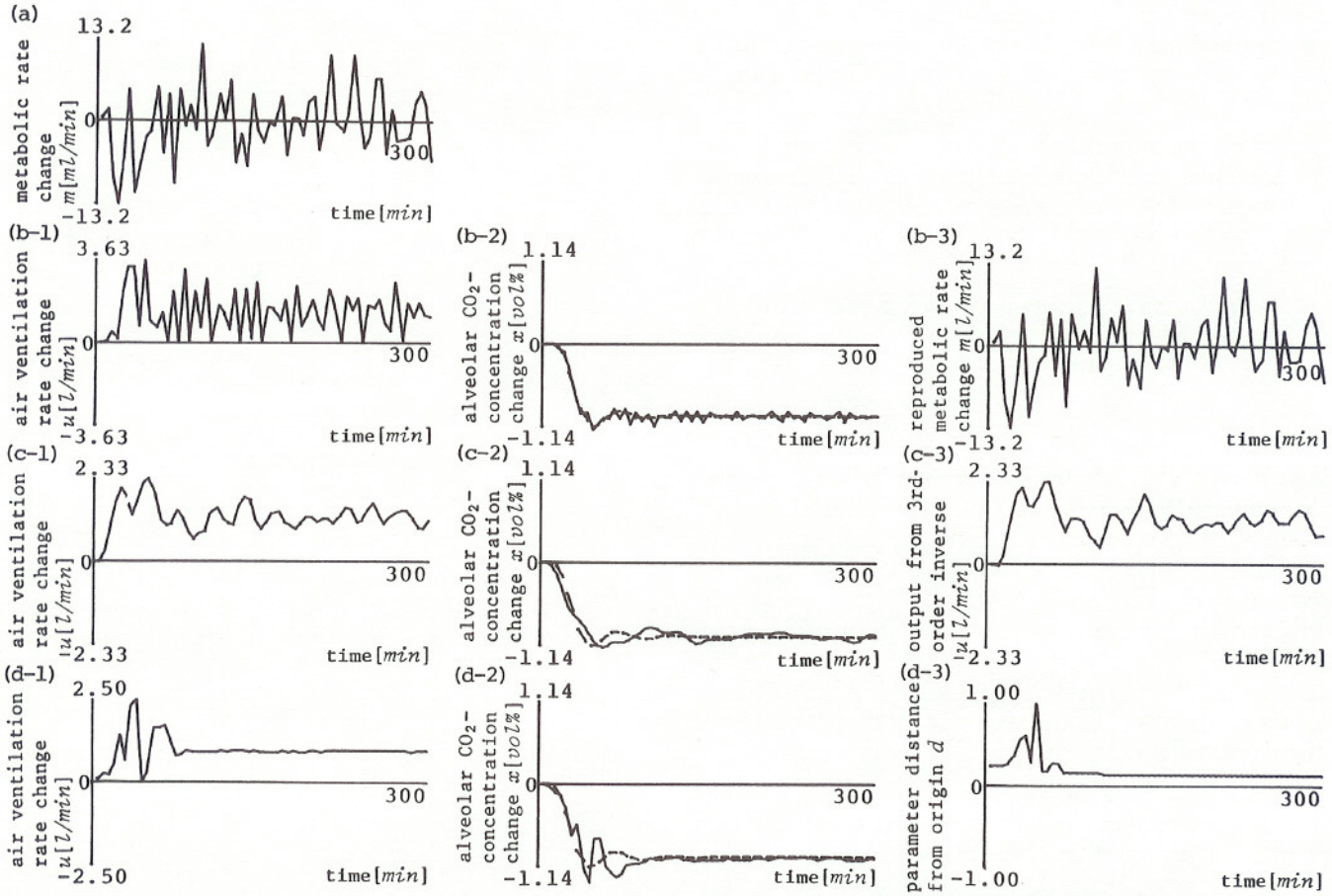

Fig 4 Dynamic characteristics of control systems of respiration for the change in desired value under the existence of metabolic rate change (τ=5.0 [min], σ=20.0 [min]). (a) Given disturbance: metabolic rate change M_p[ml/min]. (b) Control system with reproduction of metabolic rate change (μ=4.98 [%]). (c) Control system with feedback compensation (K=−7.0 [l/vol%], μ=6.76 [%]). (d) Control system with adaptive control method (μ=7.05 [%]). (b-1), (c-1), (d-1) Controlling input: air ventilation rate change U_p[l/min]. (b-2), (c-2), (d-2) Controlled output: alveolar CO_2 concentration change X_p [vol %] (solid line) with reference output X_p^M [vol %] (broken line), which is given by equation (2) for the step change from 0 to −1.0 [vol %] in desired value. (b-3) Reproduced metabolic rate change M_p [ml/min]. (c-3) Output from 3rd-order inverse U_p [l/min]. (d-3) Deviation of normalised parameter distance from θ=0.

be described by equation (17), which is here regarded as an actual respiratory system:

$$dx/dt=-\{(Q\gamma+u^0)/V_1\}x+(Q/V_1)y$$
$$-(1/V_1)ux-\{m^0/(u^0V_1)\}u \quad \ldots (17.1)$$
$$dy/dt=(Q\gamma/V_2)x-(Q/V_2)y+(1/V_2)m \quad \ldots (17.2)$$

The notations V_1, V_2 and Q denote the total volume of the alveoli, equivalent volume of body tissue and blood flow rate, respectively. This is a well-known mathematical model which consists of two compartments; one is the alveolar-arterial compartment and the other the venous-tissue compartment (Grodins *et al*, 1954; Mohler, 1973). That is, neural and/or endocrine effects are not considered in this model. Nevertheless, it describes the main characteristics of a respiratory system for the control of respiration. An equilibrium state of alveolar CO_2 concentration x^0 and average venous CO_2 concentration y^0 can be obtained as $x^0=F^i(CO_2)+m^0/u^0$, $y^0=\gamma x^0+m^0/Q+\eta$, where $F^i(CO_2)$ is the inspiratory CO_2 concentration and γ, η are constants which are determined by air pressure and the operating range of the CO_2 dissociation curve.

In order to use equation (17) as an actual respiratory system, standard values are given to system parameters as $V_1=2.6\times10^3$ [ml], $V_2=4.0\times10^4$ [ml], $Q=5.5\times10^3$ [ml], $F^i(CO_2)=3.95\times10^{-2}$ [vol %], γ=3.03 and η=32 [vol %]. Under these conditions, the equilibrium state is obtained as x^0=5.88 [vol %] and y^0=54.6 [vol %] for the constant inputs, $u^0=4.5\times10^3$ [ml/min] and $m^0=2.63\times10^2$ [ml/min].

6.2. Comparison of characteristics of three different control systems

Desirable characteristics can be realised according to three different methods, which is the main purpose of this study. Here, alveolar CO_2 concentration is controlled to follow-up 'desired characteristics' given as an output from a reference model* under the existence of a disturbance (irregular change of metabolic rate). The simulation experiments are carried out under the same conditions for the comparison of the dynamic characteristics of the three control systems. The change of

*This is considered to be given by the physician as time series data on the basis of clinical experience in the actual case.

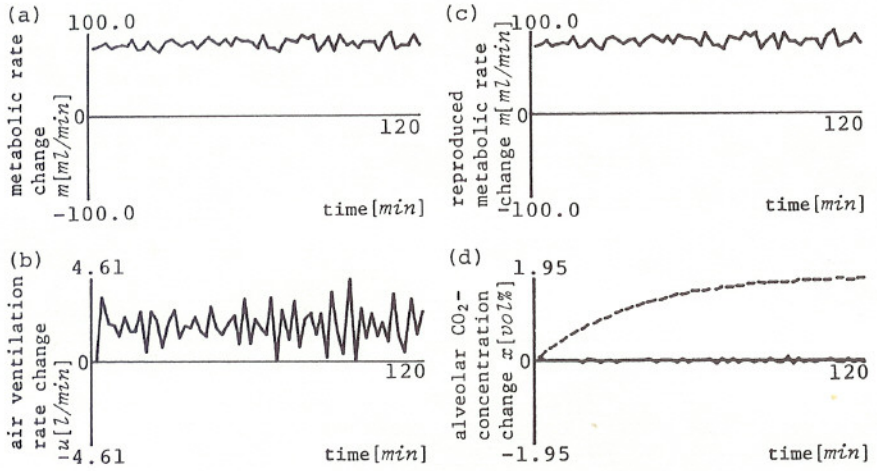

Fig 5 Dynamic characteristics of a control system by the input-reproduction method for the step-like and random metabolic rate change (r=0 [vol %], τ=2.0 [min], ϱ=1.87 [%]). (a) Given disturbance: metabolic rate change M_p [ml/min]. (b) Controlling input: air ventilation rate change U_p [l/min]. (c) Reproduced metabolic rate change M_p [ml/min]. (d) Controlled output: alveolar CO_2 concentration change $X_{c,p}$ [vol %] (solid line) with non-controlled output $X_{n,p}$ [vol %] (broken line).

metabolic rate is here assumed to take a pseudo-random value within about 5% of the value m^0*, which gives an equilibrium state with ventilation rate u^0. As m^0 has been assumed at 2.63×10^2 [ml/min], the maximum value of its change is given as about 13 [ml/min]. The reference model is adopted from equation (2), which is characterised by recommended parameters with $\beta_j=0$ ($j\geqq5$). For comparison of the efficiency of the control systems, the following evaluation function is introduced:

$$\mu=\sum_{p=1}^{n}|X_{c,p}-X_p^M|\Big/\sum_{p=1}^{n}|X_p^M| \qquad \ldots(18)$$

where n indicates the time $t=n\tau$.

Fig 4 illustrates the summarised results of simulation experiments, which were carried out for the three different control systems. Fig. 4(a) shows a given change of metabolic rate as a disturbance. The result for the control system using the input-reproduction method is shown in Fig 4(b). Fig 4(b–1) represents ventilation rate change as a controlling input†. In Fig 4(b–2), controlled alveolar CO_2 concentration change is shown by the solid line with the reference of a desired output indicated by the broken line. The result implies that the alveolar CO_2 concentration was controlled with sufficient accuracy and quickness. In addition, Fig 4(b–3) shows that the change of metabolic rate was accurately reproduced by the input-observer. It shows that the inverse system functions properly as an input-observer.

A control system by the partial model matching method is synthesised after its transformation into discrete-time representation, so that the smallest possible σ and largest possible absolute value of K are determined under the condition that its response would not fall into disorder‡. Fig 4(c) illustrates the dynamic characteristics of the designed control system for the same conditions as in the previous case. Fig 4(c–1) describes the controlling input, and Fig 4(c–2) represents the controlled alveolar CO_2 concentration with the desired characteristics given as a step response of the reference model. It is noted that the function of the 3rd-order inverse is guaranteed for the reproduction of input to the controlled object. This can be verified by the reproduction of input through 3rd-order inverse without any disturbance or with small metabolic rate change in a simulation experiment. The aspect of reproducing input series in the latter case is depicted in Fig 4(c–3).

The dynamic characteristics of the control system obtained by the adaptive control method is shown in Fig 4(d). The control system is synthesised under the same conditions as in the previous two cases. Fig 4(d–1) describes the controlling input, and Fig 4(d–2) represents the controlled alveolar CO_2 concentration with the same desired characteristics. Fig 4(d–3) indicates the aspect of convergence of estimated parameter distance.

In order to compare the efficiency of the three control systems, the evaluation values have been calculated using equation (18), which result in 4.98% for the input-reproduction method, 6.76% for the partial model matching method and 7.05% for the adaptive control method.

6.3. Characteristics of control systems as a regulator

The regulation problem maintaining an equilibrium state is the frequently encountered control demand of respiration when a step-like change in metabolic rate occurs in the respiratory system.

The change of metabolic rate is assumed to take a step-like change with a pseudo-random value within a range of about 40% of the value m^0 – ie, the maximum value of its change is given as about 1.0×10^2 [ml/min]. For the evaluation of efficiency of the control systems, the following evaluation function is used:

$$\varrho=\sum_{p=1}^{n}|X_{c,p}-r|\Big/\sum_{p=1}^{n}|X_{n,p}-r| \qquad \ldots(19)$$

where n indicates the time $t=n\tau$ and r is a desired value.

Simulation experiments for the control of the state to equilibrium were carried out by the three different control systems. The result for the control system by the input-reproduction method is shown in Fig 5. Fig 5(a) shows a given change of metabolic rate as a disturbance with which reproduced values illustrated in Fig 5(c) can be compared at every sampling time. Fig 5(b) represents ventilation rate as a controlling input. The controlled

*This kind of data sequence is adopted for simulation experiments, because any other appropriate signal resembling the actual metabolic rate change is difficult to synthesise.
†A controlling input is chosen as $U_p=0$, if $U_p<0$ is obtained, because a negative ventilation rate is physiologically inadequate.
‡In order to synthesise a control system, the subsystem $(W^{-1}-E)^{-1}$ is transformed into a difference equation after its state space description. The process of the design is explained in Appenix A3.

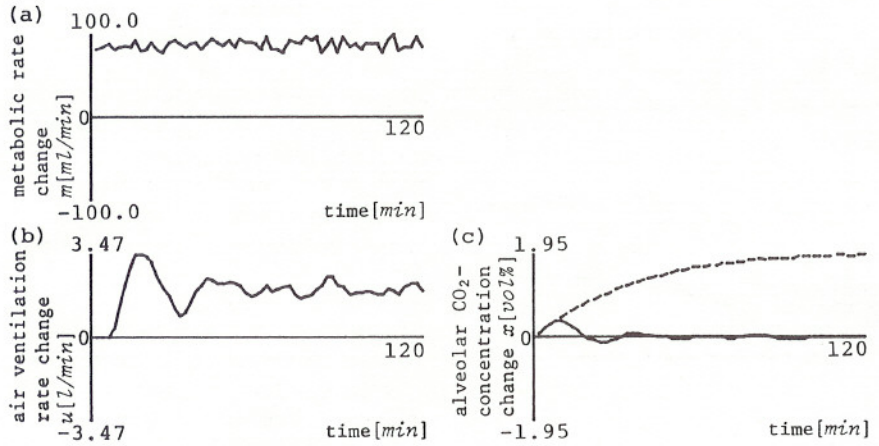

Fig 6 Dynamic characteristics of a control system by the partial model matching method for step-like and random metabolic rate change (r=0 [vol %], τ=2.0 [min], σ=6.0 [min], K=−50.0 [l/vol %], ϱ=4.37 [%]). (a) Given disturbance: metabolic rate change M_p[ml/min]. (b) Controlling input: air ventilation rate change U_p [l/min]. (c) Controlled output: alveolar CO_2 concentration change $X_{c,p}$ [vol %] (solid line) with non-controlled output $X_{n,p}$ [vol %] (broken line).

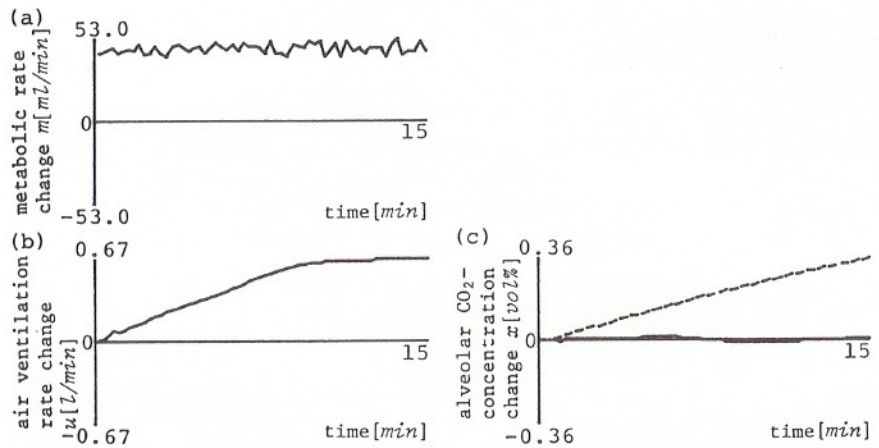

Fig 7 Dynamic characteristics of a control system by the adaptive method for step-like and random metabolic rate change (r=0 [vol %], τ=0.25 [min], ϱ=3.99 [%]). (a) Given disturbance: metabolic rate change M_p[ml/min]. (b) Controlling input: air ventilation rate change U_p [l/min]. (c) Controlled output: alveolar CO_2 concentration change $X_{c,p}$ [vol %] (solid line) with non-controlled output $X_{n,p}$ [vol %] (broken line).

alveolar CO_2 concentration change is shown in Fig 5(d). It is obvious in the input-reproduction method that the change in alveolar CO_2 concentration is maintained at zero very accurately. The dynamics of the control system designed by the partial model matching method are depicted in Fig 6. Fig 6(a) represents a change of metabolic rate given as disturbance. The ventilation rate as a controlling input is illustrated in Fig 6(b) and the controlled alveolar CO_2 concentration change is shown in Fig 6(c). Considering the restriction of choosing parameters in the design of the control system, as seen in equation (8), σ cannot be selected extremely small for the regulation in this case. Therefore, the dynamic characteristics are not as good as in the previous case. However, the control is performed satisfactorily for sufficiently small σ. In the case of the adaptive control system, a similar result as in the case of the input-reproduction method is obtained with almost zero desired value*, provided that the sampling interval is selected appropriately. It is necessary to choose a short interval sampling time τ in order to regulate the state in a short time. The desirable dynamics as a regulator can be seen in Fig 7, which results from the control system with sampling interval τ=0.25 [min]. The illustrations of Fig 7 are arranged as in Fig 6.

7. Discussion of characteristics of control systems

7.1. Amount of knowledge about the controlled object

The synthesis of a control system with less knowledge of the controlled object is required in practice. On the basis of such a concept, control systems have been synthesised in this study according to the amount of knowledge obtainable from the controlled object and to the clinical demands of accuracy of control performance.

The first method is for an accurate control whereas the second and third are not for an accurate control in view of the amount of knowledge about the respiratory regulation system. However, the first control method is difficult to apply directly to an actual case, unless the parameters have been obtained beforehand. On the other hand, considerable accurate and robust control is possible by the second method using average parameters or roughly identified parameters because the design parameters can be adjusted accordingly with more or less recognition of the controlled object†. The third method is for control when the parameters cannot be obtained or when only their approximate values can be obtained on the rough assumption of the structure of the respiratory system. This enables us to control respiration to have an appropriate value or to show the desirable dynamic characteristics specified beforehand by the

*It is in practice not good for the calculation of adaptive controlling input, if a desired value is always exactly zero.

†The characteristics of the control system can be compared with those of the one based on the more (less) precise description of the controlled object. The design-parameter σ can be made smaller (greater) in the latter case than in the former for the same step change in desired value. That is, more (less) precise recognition and description of the controlled object basically provides better (worse) parameters, which offer more (less) rapid response under the same design condition (Wakamatsu and Kitamori, 1984).

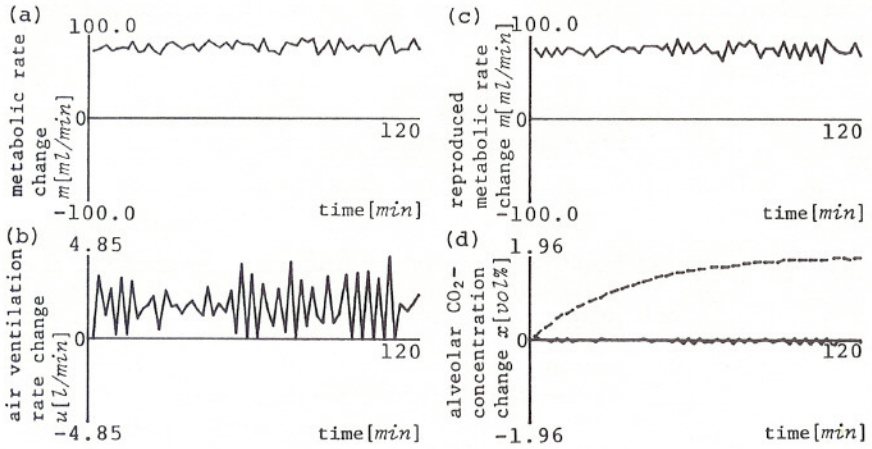

Fig 8 Inspection of the robustness of a control system by the input-reproduction method for step-like and random metabolic rate change (r=0 [vol %], τ=2.0 [min] ϱ=2.47 [%]). Parameter Q increases by 20 [%]. (a) Given disturbance: metabolic rate change M_p[ml/min]. (b) Controlling input: air ventilation rate change U_p [l/min]. (c) Reproduced metabolic rate change M_p [ml/min]. (d) Controlled output: alveolar CO_2 concentration change $X_{c,p}$ [vol %] (solid line) with non-controlled output $X_{n,p}$ [vol %] (broken line).

physician, when respiration must be artificially controlled in emergency clinics and there is insufficient knowledge of the dynamics of a patient. Thus, it can be emphasised that the proposed control methods can be applied to actual cases by combinations of control systems designed with the reverse order of gradual increment in the amount of knowledge about the respiratory system.

7.2. Limitation of control systems

A control requirement for accurate and quick response generally contradicts that of robustness within the same control system. It is thus necessary to evaluate the overall technical efficiency of a control system, as there might be concern that the merit of accuracy and short response-time cannot be maintained in practical use with slight deviations of system parameters caused by some misunderstandings of the controlled object which result from the measurement error of system parameters and/or their chronic change.

The dynamics of the first control system had been expected to be most seriously affected by parameter change. The parameters V_2 and η physiologically change little even if the metabolic rate changes. However, V_1 and Q increase when metabolic rate increases. The effects of 20% deviation of parameters V_1 and Q have been examined in the case of a regulation problem using an evaluation function given by equation (19). The control efficiencies obtained from the simulation experiments for the change in V_1 and Q are ϱ=1.88 [%] and ϱ=2.47 [%]*, respectively. The dynamics for the change in parameter Q are illustrated in Fig 8, which can be compared with the result given in Fig 5. It shows that a robustness of the control system can be largely maintained against parameter change. It is noted that the approximation given by equation (3) can be held only for the physiologically appropriate amount of ventilation rate change. That is, the accuracy of the control system cannot be expected by the input-reproduction method, as equation (3) cannot represent any exact respiratory system for a larger ventilation rate change.

For the second control system, the amplitude of input also comes into question because the inverse has a restriction of its input amplitude for application to the control system. That is, the inverse truncated by finite terms may not adequately perform its function with a large input, even if the controlled object is perfectly described. In addition, there may appear some effect due to unrecognisable parts of the controlled object. In other words, the parameters K and σ should be selected so as to give an input with an appropriate amplitude, which satisfies the range of convergence of an inverse system (Schetzen, 1976). It should be particularly noted in the design of control systems for the great change in desired value.

In the third case, parameters are automatically estimated† and a controlling input is generated using the estimated parameters. Therefore, there is no amplitude restriction, which is a weak point in the previous two control systems. However, the internal states of the respiratory system have been assumed to be bounded because of the non-linear characteristics of the living organism. It is therefore necessary to inquire whether or not internal states are bounded, which is important for controlling the respiratory system satisfactorily using a directly calculated controlling input, even if the safety of life can be guaranteed because of the above mentioned non-linearity.

8. Conclusions

The methods of controlling a respiratory system have been discussed on the basis of the system representation by discrete-time Volterra series, by which only the input–output relation can be described but internal states are not recognised. The main purpose of the study has been to investigate whether or not the methods are applicable to the living organism such as the respiratory system with ambiguous structure and time variant parameters. Thus, an attempt has been made to examine the control efficiency of three different control systems, which follow-up a desired value under circumstances of persistent change of metabolic rate.

The last method is suitable for the synthesis of a control system using ambiguous and/or time varying parameters from the viewpoint of the control principle

*This suggests that the blood flow rate should not be treated as a parameter but as an independent state variable given its large range of deviation, as far as the input-reproduction method is concerned (Wakamatsu *et al*, 1981; Wakamatsu, 1984).
†Real parameter values cannot always be guaranteed to be estimated. Only the output error converges to zero.

with appropriate selection of gain Γ_p. In other words, it can be applied to the patient who needs immediate artificial respiration without any identification of dynamic characteristics. It is also useful for medical care in the absence of doctors, because it has a facility to control automatically the dynamics of an alveolar CO_2 concentration according to the desired characteristic curve, which has been given as an average data of clinical experiences. The second one is useful when important parameters are known beforehand. The first method should be applied to help autonomic respiration, when accurate control of arterial CO_2 concentration is required under the use of muscle relaxant and/or open chest surgery.

Thus, the methods can provide high quality of control of respiration for various clinical situations by appropriate combination, although the first two control systems cannot avoid having an inferior performance resulting from the different dynamics according to differences between individuals. Those are very significant for the application of the proposed control methods to clinics, because it is in principle possible to synthesise a control system with desirable dynamics and because it is not necessary to acquire parameters of individuals, which is time-consuming and laborious work for medical staff. It is thus concluded that the main purpose of this study – ie, the investigation of the possibility of the actual application of these new control methods – was attained, it being noted that these types of control systems are useful even for other biological systems whose input–output relations are not exactly known and whose parameters chronically change to some extent.

Acknowledgements

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APPENDICES

A.1 Input-observer for input-reproduction method

The input-observer given by equation (4) is characterised by the following coefficients:

$$a_p = h_0^m + h_{00}^{um} U_p + h_{000}^{uum} (U_p)^2 \quad \ldots \text{(a.1.1)}$$

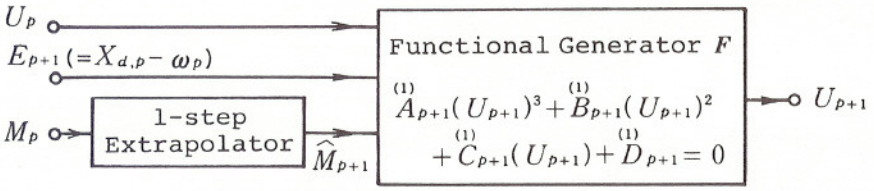

Fig A-1 Structural description of dynamical compensator (discrete-time description).

$$b_p=v_{p-1}+X_{u,p}-X_p \qquad \ldots \text{(a.1.2)}$$

$$v_{p-1}=\sum_{i=1}^{p} h_i^m M_{p-i}+\sum_{j=1}^{p} h_{0j}^{um} U_p M_{p-j}$$
$$+\sum_{k=1}^{p} h_{00k}^{uum} U_p U_p M_{p-k}+\sum_{i=1}^{p}\sum_{j=i}^{p} h_{ij}^{um}$$
$$\times U_{p-i} M_{p-j}+\sum_{j=1}^{p}\sum_{k=j}^{p} h_{0jk}^{umm} U_p U_{p-j} M_{p-k}$$
$$+\sum_{i=1}^{p}\sum_{j=i}^{p}\sum_{k=j}^{p} h_{ijk}^{uum} U_{p-i} U_{p-j} M_{p-k}$$

X_p is the value of alveolar CO_2 concentration and $X_{u,p}$ is the value of X_p excluding the effect of change of metabolic rate from equation (3).

A.2 Dynamical compensator for input-reproduction method

The dynamical compensator is a controller which consists of a 1-point extrapolator and a functional generator as illustrated in Fig A-1. The functional generator given by equation (5) is precisely described as follows:

$$A_p=h_{000}^{uuu} \qquad \ldots \text{(a.2.1)}$$

$$B_p=h_{00}^{uu}+h_{000}^{uum}\hat{M}_p+\sum_{k=1}^{p} h_{00k}^{uuu} U_{p-k}+\sum_{k=1}^{p} h_{00k}^{uum} M_{p-k} \qquad \ldots \text{(a.2.2)}$$

$$C_p=h_0^u+h_{00}^{um}\hat{M}_p+\sum_{j=1}^{p} h_{0j}^{uu} U_{p-j}$$
$$+\sum_{j=1}^{p} h_{0j}^{um} M_{p-j}+\sum_{j=1}^{p}\sum_{k=j}^{p} h_{0jk}^{uuu} U_{p-j} U_{p-k}$$
$$+\sum_{j=1}^{p}\sum_{k=j}^{p} h_{0jk}^{uum} U_{p-j} M_{p-k} \qquad \ldots \text{(a.2.3)}$$

$$D_p=-E_p+h_0^m\hat{M}_p=-(X_{d,p}-\omega_{p-1})+h_0^m\hat{M}_p \qquad \ldots \text{(a.2.4)}$$

where $X_{d,p}$ is a desired value r at sampling time p. It is noted that ω_{p-1} corresponds to the value of equation (3) in which summation is carried out from $i=1$ instead of $i=0$. That is, ω_{p-1} is the contribution of past input series through the previous sampling time to the present output. ω_{p-1} is conveniently represented by controlled value x in Fig 1.

A.3 Calculation of controlling input by partial model matching

In order to obtain discrete-time controlling input from equation (7), the following calculations are carried out as for the components v and v_f:

The integration is performed using $D=(1-z^{-1})/\tau$ as an operator for the calculation of component $v=KD^{-1}(e)$ in discrete-time. In order to calculate the component $v_f=KD^{-1}(W^{-1}-E)(x)=K(\beta_1\sigma+\beta_2\sigma^2 D+\beta_3\sigma^3 D^2+\ldots)(x)$, a four-point algorithm, $D=(1+3z^{-1}-3z^{-2}-z^{-3})/6\tau$, is adopted for the smooth differential operation.

σ is determined to give an optimal evaluation value with its gradual decrement and K is determined with gradual increment of its absolute value (Wakamatsu, 1985b).

In this study, the relation

$$|K|=-a\sigma+bK_0(a, b, K_0>0) \qquad \ldots \text{(a.3.1)}$$

has been adopted, considering that the response is likely to fall into a disorder with a smaller σ, but that absolute value of gain K can be made larger with a smaller σ.

A.4 Parametric representation of the system for adaptive control

If parameter vector θ_0 is defined as

$$\theta_0^T=[h_1^u, h_2^u, h_{11}^{uu}, h_{12}^{uu}, h_{22}^{uu}, h_{111}^{uuu}, h_{112}^{uuu}, h_{122}^{uuu}, h_{222}^{uuu}] \qquad \ldots \text{(a.4.1)}$$

and if state vector $\xi_{0,p-1}$ is constructed from the input data through previous sampling time as

$$\xi_{0,p-1}^T=[U_{p-1}, U_{p-2}, U_{p-1}U_{p-1}, U_{p-1}U_{p-2}, U_{p-2}U_{p-2},$$
$$U_{p-1}U_{p-1}U_{p-1}, U_{p-1}U_{p-1}U_{p-2}, U_{p-1}U_{p-2}U_{p-2},$$
$$U_{p-2}U_{p-2}U_{p-2}] \qquad \ldots \text{(a.4.2)}$$

equation (10) is transformed into parametric representation given by equation (a.4.3.).

$$X_p=\alpha_3(U_p)^3+\alpha_2(U_p)^2+\alpha_1(U_p)+\theta_0^T\xi_{0,p-1} \qquad \ldots \text{(a.4.3)}$$

where

$$\alpha_1=h_0^u+\theta_1^T\xi_{1,p-1}$$
$$\alpha_2=h_{00}^{uu}+\theta_2^T\xi_{2,p-1},$$
$$\alpha_3=h_{000}^{uuu},$$
$$\xi_{1,p-1}^T=[U_{p-1}, U_{p-2}, U_{p-1}U_{p-1}, U_{p-1}U_{p-2}, U_{p-2}U_{p-2}],$$
$$\xi_{2,p-1}^T=[U_{p-1}, U_{p-2}]$$